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Dynamics of flux tubes in large N gauge theories

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Abstract: The gluonic field created by a static quark anti-quark pair is described via the AdS/CFT correspondence by a string connecting the pair which is located on the boundary of AdS. Thus the gluonic field in a strongly coupled large N CFT has a stringy spectrum of excitations. We trace the stability of these excitations to a combination of large N suppressions and energy conservation. Comparison of the physics of the $N = \infty$ flux tube in the $\mathcal{N} = 4$ SYM theory at weak and strong coupling shows that the excitations are present only above a certain critical coupling. The density of states of a highly excited string with a fold reaching towards the horizon of AdS is in exact agreement at strong coupling with that of the near-threshold states found in a ladder diagram model of the weak-strong coupling transition. We also study large distance correlations of local operators with a Wilson loop, and show that the fall off at weak coupling and $N = \infty$ (i.e. strictly planar diagrams) matches the strong coupling predictions given by the AdS/CFT correspondence, rather than those of a weakly coupled $U(1)$ gauge theory.

Keywords: AdS-CFT Correspondence, 1/N Expansion.

Contents

1. Introduction

The usual picture of quark confinement is that the color flux lines between the quark and antiquark do not spread throughout space, as they do in electromagnetism, but rather form a narrow tube. Furthermore, this flux tube ("QCD string") is supposed to have its own dynamics, since it can support transverse oscillations. In fact, the hadron spectrum consists of narrow resonances which may be associated with the excitations of this flux tube. In the 't Hooft large N limit [1] the self-interactions of the QCD string vanish, and we expect the gauge theory to have a dual description in terms of string theory with coupling $O(1/N)$. The string dynamics should describe all excitations of the gauge theory flux tube. Since these arguments are independent of whether the theory is confining or not, we expect to have a string description even for non-confining theories.

The AdS/CFT correspondence provides such a description; for example, the string dual of the $\mathcal{N} = 4$ $SU(N)$ super Yang Mills theory is type IIB string theory on $AdS_5 \times S^5$ [2]. This theory is conformal and hence non-confining. Nevertheless, the potential between a fixed quark and a fixed anti-quark can be computed, at infinite N and large $\hat{\lambda} = g^2N/4\pi^2$, using the semiclassical string description [3]. The resulting potential goes like $-\sqrt{\hat{\lambda}}/L$, similar to the weakly coupled gauge theory potential $-\hat{\lambda}\pi/L$. Of course, this Coulombic form of the potential is a consequence of the conformal symmetry. The fact that the L

dependence is the same should not obscure the fact that the dynamics of the flux tube at strong coupling is dramatically different than at weak coupling. In fact, the string in AdS_5 has for large $\hat{\lambda}$ a rich spectrum of discrete energy levels between $E \sim -\sqrt{\hat{\lambda}}/L$ and $E = 0$, reflecting the many degrees of freedom of a fluctuating flux tube [4] (see also [23]). On the other hand, we will show that at weak coupling there is only an empty energy gap between $E = -\pi\hat{\lambda}/L$ and $E = 0$. Thus there must be a critical coupling $\hat{\lambda}_c$ above which these excitations begin to appear.

A simple way to see some indication of such a transition as a function of the coupling is to consider the Klein-Gordon (or Dirac) equation for a particle in the presence of a Coulomb potential. These equations have a "fall to the center" instability when the coupling is larger than a critical value of order one. The linearized Yang-Mills equations exhibit a similar behavior [5].

To explore the weak/strong coupling transition at large N, we consider a soluble model based on a truncated subset of planar Feynman diagrams, the ladder graphs in Feynman gauge [6]. Although this model gives an uncontrolled approximation, we will find that it captures a surprising number of the expected features of the full answer. For instance, there is a critical coupling where the number of discrete states of the flux tube jumps from one to infinity. The quark anti-quark ground state energy has the form $E_0 = -f(g^2N)/L$. If we supply some energy $\Delta E > |E_0|$ then the quarks can become unbound, which is to say that the system energy becomes continuous. We will say in this situation that the system ionizes and that $E = 0$ is the ionization threshold. The rich spectrum of bound states mentioned above arises when the total energy of the system is less than zero. Of particular interest are some discrete states with energies accumulating at the ionization threshold. At strong coupling these states are described by folded strings moving along the radial direction of AdS_5 with some folds approaching the horizon. These states have similar properties to the near threshold states appearing in the ladder model above a critical coupling. In fact we will find a precise matching between the density of states in the ladder approximation and the density of such string states containing one fold. The existence of this large number of near threshold states has interesting consequences for the renormalization properties of adjoint Wilson loops.

Another set of observables that shows an apparently new qualitative behavior at strong coupling [4] are the one point functions of local operators in the presence of Wilson loops. In this case, however, we will use planarity and energetic considerations to show that the qualitative behavior at large distances is actually the same at weak and strong coupling.

2. Flux tube energy spectrum

In this paper we consider mainly the $\mathcal{N} = 4$ super Yang Mills theory with fixed static quark and anti-quark sources. They are represented by two antiparallel time-like Wilson lines separated by a distance L. We also have to specify the locations on the S^5 for each of these lines, which correspond to the orientations of their scalar charges. We mainly choose these locations to be the same, but will also comment on the situation when they are different.

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2.1 Weak coupling analysis

In Coulomb gauge, transverse gluons decouple from the Wilson lines, so the lowest order system energy is trivially the Coulomb potential

$$E_0^{\text{singlet}} = -\frac{g^2(N^2-1)}{8\pi N}\frac{1}{L} \to -\frac{\pi\hat{\lambda}}{2L} \tag{2.1}$$

$$E_0^{\text{adjoint}} = +\frac{g^2}{8\pi N}\frac{1}{L} \to 0 \tag{2.2}$$

where the arrows show the $N \to \infty$ limits. For the $\mathcal{N} = 4$ case, $E_0^{\text{singlet}} = -\pi\hat{\lambda}/L$, because of scalar exchange [7, 8].

At weak coupling, $\hat{\lambda} \ll 1$, we do not expect to find any other discrete states with energy $E_0^{\text{singlet}} < E < 0$. One could imagine a state consisting of an extra gluon that is interacting with the quark and antiquark through planar interactions. However a simple argument based on the uncertainty principle shows that such a state has $E > 0$. A massless particle localized within distance r will have kinetic energy $\sim 1/r$ and potential energy $\sim -g^2N/r$. The kinetic energy is bigger at weak coupling, so we do not expect a bound state.[1]

However, there do exist states with unbound gluons alongside the singlet $q\bar{q}$ system with continuous total energy $E \geq E_0^{\text{singlet}}$. This leads to the following interesting effect. In analogy with QED we would have expected that if we put the system in its lowest energy state (the singlet) and shake one of the quarks a little bit, the system would decay immediately by emitting massless radiation, since the theory has no gap. And, indeed, this is what happens at finite N. However, if the injected energy $\Delta E < |E_0^{\text{singlet}}|$ the final state of the quark anti-quark system must by energy conservation be the color singlet ground state, which in turn implies that the emitted gluons must be in an overall color singlet. But color singlet glue emission is suppressed by powers of $1/N$, whether or not the gluons are bound into glueballs. Thus we conclude that at weak coupling and $N = \infty$, the energy spectrum of the quark anti-quark system consists of a single discrete level E_0^{singlet} separated by a gap from a continuous spectrum $E \geq 0$. At weak coupling we must think of the $N = \infty$ "flux tube" connecting the quark to the antiquark as spread out all through space (somewhat similar to the electric field of a dipole in QED) but with a certain "stiffness" characterized by this energy gap (quite unlike QED). But at weak coupling this stiffness is too fragile to support vibrations. But we can imagine that as the coupling increases and the gap widens, keeping $N = \infty$, this stiffness becomes more robust eventually supporting vibrations, populating the gap with discrete levels.

2.2 Wilson loop at weak coupling

In order to think in a clear way about the interpolation from weak to strong coupling, it is useful to understand the results of the previous subsection in terms of summing diagrams at large N. To keep track of $1/N$ suppressions it is convenient to calculate a single trace

[1]One might wonder if there could be states with energies $-(g^2N)^n/L$, with $n > 1$ which, at weak coupling could be confused with the continuum. However the argument based on the uncertainty principle rules them out.

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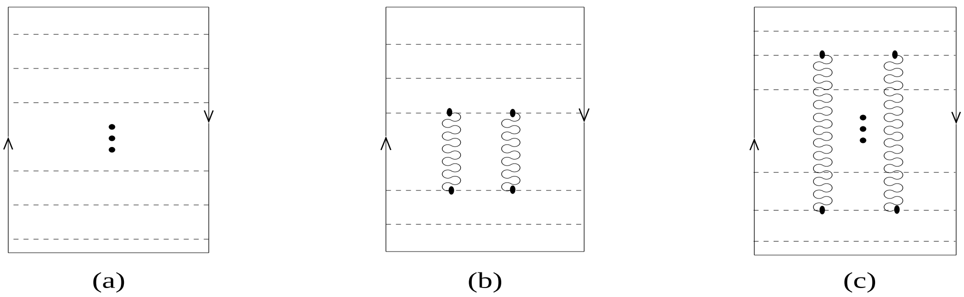

Figure 1: Some Coulomb gauge Feynman diagrams contributing to the Wilson loop. The wavy lines represent transverse gluons and the dashed lines instantaneous Coulomb exchange. The Coulomb exchange diagrams (a) sum up to the ground state pole in $R(E)$. The planar diagram (b) gives a contribution to the continuum cut $E \geq 0$. The gap destroying nonplanar diagram (c) is suppressed at $N = \infty$.

gauge singlet, and the Wilson loop $W(L, T)$ for an $L \times T$ rectangle fits the bill perfectly. The energy spectrum of the flux tube can be read off from the exponential time dependence of the expectation of the Wilson loop $W(L, T)$ for an $L \times T$ rectangle.

$$\langle W(L,T) \rangle = \sum_i \rho_i e^{-TE_i} \tag{2.3}$$

or equivalently from the singularities in E of its Laplace transform, the resolvent

$$R(E) \equiv \int_0^\infty dT e^{ET} \langle W(L,T) \rangle = \sum_i \frac{\rho_i}{E_i - E} \,. \tag{2.4}$$

Discrete eigenstates produce simple poles in R and the continuum produces cuts (implying imaginary parts) in R. The optical theorem relates the imaginary part of R to the decay rate of the system.

Then the results of the previous subsection can be succinctly summarized by writing, for $\hat{\lambda} \ll 1$,

$$R(E) \approx \frac{\rho_0}{-\pi\hat{\lambda}/L - E} + \int_0^\infty dE' \frac{\rho(E')}{E' - E} + \frac{1}{N^2} \int_{-\pi\hat{\lambda}/L}^\infty dE' \frac{\rho_1(E')}{(E' - E)} \tag{2.5}$$

where $\rho_0 = O(1)$, $\rho(E') = O(\hat{\lambda})$, and $\rho_1(E') = O(\hat{\lambda}^2)$. The $N \to \infty$ limit removes the gap destroying third term. We used the ground state energy of the $\mathcal{N} = 4$ theory. For pure Yang-Mills, just substitute $\hat{\lambda} \to \hat{\lambda}/2$. In figure 1 we display a sample Coulomb gauge Feynman diagram contributing to each of the terms. [2] By focusing on the infinite N Wilson loop, we cleanly remove the gap destroying processes that are present at finite N.

The Wilson loop also has a transparent interpretation on the string side of the duality, which makes it a particularly apt observable for following the weak/strong coupling interpolation. In the next section we discuss the flux tube spectrum at strong coupling. We will see that the main qualitative change is the replacement of the single pole term with an infinite number of pole terms accumulating at $E = 0$.

[2] We chose diagrams with two transverse gluons because their color states are more generic than a single gluon which only exists in an adjoint representation.

2.3 Strong coupling analysis

At strong coupling we can study the system using the AdS/CFT correspondence [2]. The $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory is dual to type IIB superstring theory on the manifold $\mathrm{AdS}_5 \times \mathrm{S}_5$ whose metric is

$$ds^2 = \frac{R^2}{z^2}(dz^2 + dx_\mu^2) + R^2 d\Omega_5^2 \,, \tag{2.6}$$

and $R^2 = \sqrt{g^2 N} \alpha'$. In the large N limit the superstrings are noninteracting, just as glueballs would be in a confining gauge theory. Configurations with external quarks were studied in [3]. The absence of confinement implies that the energy of an isolated quark has no IR divergence. The dual of a quark at spatial position $\vec{x}$ in the gauge theory is a type IIB string stretched along the z direction all the way to infinite z [3]. The energy of such a string is

$$\frac{R^2}{2\pi\alpha'} \int_\epsilon^\infty \frac{dz}{z^2} \,, \tag{2.7}$$

where we have introduced a UV cut-off at small $z = \epsilon$. The IR corresponds to the large z region and we see that (2.7) is finite there.

If we consider a quark at $x_1 = -L/2$ and an anti-quark at $x_1 = L/2$, whose colors are different, they cannot form a color singlet bound state. The corresponding state in AdS space has two straight strings, one running from $z = \epsilon$ to ∞ at $x_1 = -L/2$, and the other running from $z = \infty$ to ϵ at $x_1 = L/2$. If the colors are correlated so that a $q\bar{q}$ colorless bound state can form, then the dual configuration is given by a single segment of the string (see figure 2(a)) whose position $z_c(x)$ was determined in [3]; it satisfies

$$\left(z_c'(x)\right)^2 = \frac{z_m^4}{z_c^4(x)} - 1 \,. \tag{2.8}$$

The maximum value of the z-coordinate is attained at the mid-point

$$z_m = z_c(0) = \frac{\Gamma(1/4)^2 L}{(2\pi)^{3/2}} \,. \tag{2.9}$$

This string is the dual of the gluonic field that forms between the static quark and anti-quark in the conformal gauge theory. The string energy has the separation dependence of a Coulomb potential [3]

$$E_0(L) = -c\frac{\sqrt{g^2 N}}{L} \,, \qquad c = \frac{4\pi^2}{\Gamma(\frac{1}{4})^4} \,. \tag{2.10}$$

Callan and Guijosa [4] studied small oscillations about the static string connecting the heavy quark and anti-quark (see also [23]). The equation for the oscillation modes transverse to the z direction, derived in [4], may be written as

$$\left[\partial_t^2 - \frac{z_c^4(x)}{z_m^4}\partial_x^2\right] y(x,t) = 0 \,. \tag{2.11}$$

After imposing the Dirichlet boundary conditions, $y(\pm L/2, t) = 0$, the exact normal mode frequencies are $\omega_n = \xi_n / z_m$, with ξ_n satisfying

$$\xi_n \sqrt{\xi_n^4 - 1} \int_0^1 \frac{t^2 dt}{[1 + \xi_n^2 t^2]\sqrt{1 - t^4}} = \frac{n\pi}{2}, \qquad n = 1, 2, \ldots \tag{2.12}$$

This result follows from the quantization of the phase in [4, eq. (33)] and corrects a typo in eq. (38) of that reference. The integral in (2.12) is elliptic, and the equation must be solved numerically. However, for large frequency it becomes elementary [9]:

$$\xi_n \sim \frac{(n+1)\pi}{2} \left[\int_0^1 \frac{dt}{\sqrt{1 - t^4}} \right]^{-1} = (n+1) \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad n \gg 1\,. \tag{2.13}$$

This result is equivalent to the WKB approximation applied to (2.11), but with n replaced by $n + 1$. The analysis of small transverse oscillations thus gives discrete excitation energy levels $E_{\{N_n\}} - E_0 = \sum N_n \omega_n$. This stringy spectrum is of the same qualitative nature as the spectrum around a confining flux tube of length L, which can be deduced from the Nambu-Goto action in the static gauge. Of course, in our non-confining example, this small oscillation spectrum is only valid if $E - E_0 \ll |E_0|$. If $\hat{\lambda}$ is large but finite we will need to go beyond the quadratic approximation for computing the energy when $\sum N_n \xi_n \sim \sqrt{g^2 N}$.

Now, let us also derive the fluctuation equation for the "longitudinal modes" coming from the function $z(x, t)$. In the static gauge $\tau = t$, $\sigma = x$, the Nambu-Goto action is

$$S = -\frac{R^2}{2\pi\alpha'} \int dt \int_{-L/2}^{L/2} dx \; z^{-2} \sqrt{1 - (\dot{z})^2 + (z')^2}\,. \tag{2.14}$$

From it we find the equation of motion

$$\frac{\partial^2 z}{\partial t^2} \left(1 + (z')^2\right) - \frac{\partial^2 z}{\partial x^2} \left(1 - (\dot{z})^2\right) - \frac{2}{z} \left(1 - (\dot{z})^2 + (z')^2\right) - 2 \frac{\partial^2 z}{\partial t \partial x} \dot{z} z' = 0\,. \tag{2.15}$$

Expanding around the minimal energy solution z_c as $z = z_c(x) + u(x, t)/z_c(x)^2$, we find that the linearized equation for u is

$$\left[\partial_t^2 - \frac{z_c^4(x)}{z_m^4} \partial_x^2 \right] u(x, t) = \frac{2 z_c^2(x)}{z_m^4} u(x, t)\,. \tag{2.16}$$

Compared to the transverse oscillation equation (2.11) there is an attractive potential term on the right hand side.[3] This equation has a series of eigenfrequencies $\tilde{\omega}_n = \tilde{\xi}_n / z_m$, and each such oscillator can have a discrete occupation number $\tilde{N}_n$. Thus, for the longitudinal modes we again obtain a stringy form of the spectrum. Applying the WKB approximation to (2.16), gives

$$\int_0^1 dt \frac{\sqrt{\tilde{\xi}_n^2 + 2t^2}}{\sqrt{1 - t^4}} = \frac{n\pi}{2}, \qquad n \gg 1\,. \tag{2.17}$$

[3] This equation is equivalent to the longitudinal oscillation equation obtained using lightcone quantization in [9].

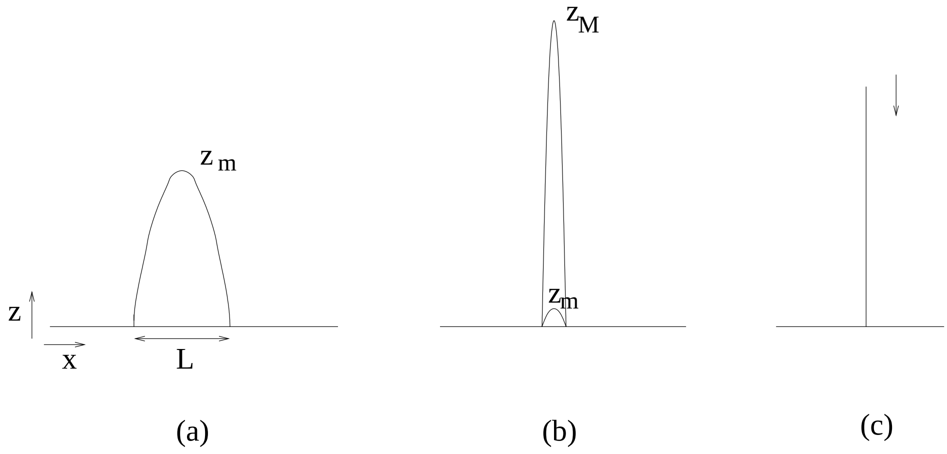

Figure 2: In (a) we show the string going between two points in the boundary which gives the dual description of the flux tube going between the quark and the anti-quark. In (b) we stretch the string up to a large $z_M \gg z_m$. In (c) we replace the stretched string in (b) by a folded string sitting at a single spatial point on the boundary.

In order to analyze the solution with high occupation number we resort to a different approximation. We consider the following thought experiment. Let us stretch the string from $z = z_m$ to $z_M \gg z_m$, see figure 2(b). Such a string will be near the ionization threshold since $0 > E \gg E_0$. In this case, we can neglect the transverse separation $L \sim z_m \ll z_M$ and consider the string as a folded string that lies at one point in x and extends purely along the z direction, see figure 2(c). The dynamics of these strings is very similar to the dynamics of folded strings in two flat dimensions studied in [10]. The spacetime picture for these folded strings is easy to understand. We simply consider the tip of the fold as a massless particle which is being pulled by a string. So we consider a folded string that is stretched along the z direction and is located at a point of the R^3 of the spatial coordinates of the boundary. We put a UV cutoff in the radial direction at $z_m \sim L$. We now stretch the string from z_m to z_M, with $z_M \gg z_m$. The energy of this stretched string is

$$E = 2\frac{R^2}{2\pi\alpha'}\int \frac{dz}{z^2} = -2\frac{R^2}{2\pi\alpha'}\frac{1}{z_M}\,, \tag{2.18}$$

where we subtracted a cut-off dependent constant in the energy in such a way that the energy becomes zero when $z_M \to \infty$. We are interested in highly excited states with energies $0 > E \gg E_0 \sim -\frac{1}{L}$.

If we stretch the folded part of the string and then release it, the tip of the string will move towards the boundary of AdS. The tip will be accelerated very quickly to nearly the speed of light. So, we can think of the tip of the string as a massless particle with momentum and energy $p^0 = |p_z|$. Energy conservation implies

$$p^0 - \frac{R^2}{\pi\alpha'}\frac{1}{z} = E\,, \qquad p^0 = |p_z| = \frac{R^2}{\pi\alpha'}\left(\frac{1}{z} - \frac{1}{z_M}\right). \tag{2.19}$$

Using the Bohr-Sommerfeld quantization condition

$$n = \frac{1}{2\pi}\oint p_z dz = \frac{R^2}{\pi^2\alpha'}\int_{z_m}^{z_M} dz\left(\frac{1}{z} - \frac{1}{z_M}\right) = \frac{R^2}{\pi^2\alpha'}\left[\log\frac{z_M}{z_m} - 1 + \frac{z_m}{z_M}\right]. \tag{2.20}$$

To leading order we have $n \sim \log z_M/z_m$. Combining this with (2.18) we find that

$$-\log(-Ez_m) \sim \frac{\pi^2\alpha'}{R^2} n \sim \frac{\pi^2}{\sqrt{g^2N}} n \tag{2.21}$$

where we neglected a constant piece which could depend on the details of the regularization procedure. We can think of (2.21) as giving us a density of states $dn/d\log|E|$. The conformal symmetry determines (2.21) up to a function of the coupling. Thus, the AdS geometry was used to compute the square root appearing in (2.21). Note that this density of states is also related to the scattering phase shift accumulated when the tip of the string moves from the boundary to the region near the horizon and back.

3. Comparison with the ladder appoximation

By comparing the weak coupling and strong coupling results it is clear that there should exist a critical coupling where new states appear from the continuum. To definitively analyze this issue would require a complete understanding of the sum of all planar diagrams, a hugely ambitious program beyond the scope of this paper.

In this section we will consider instead a simple, albeit uncontrolled, approximation to the gauge theory computation: the sum of ladder diagrams in Feynman gauge. This approximation was introduced in [6] where it was shown that the energy eigenvalue problem for the ladder model is equivalent to the Schroedinger-like differential equation

$$\left[-\partial_t^2 - \frac{\hat\lambda}{L^2+t^2}\right]\psi = -\frac{E^2}{4}\psi\,; \qquad \hat\lambda \equiv \frac{g^2N}{4\pi^2} \tag{3.1}$$

where the energy E is defined so that threshold is at $E = 0$. Later it was shown that the analogous approximation for circular Wilson loops [7, 8], the sum of "rainbow" graphs, is exact for $\mathcal{N} = 4$ at $N = \infty$, since all the more complicated planar graphs apparently cancel out. It was even argued in [8] that it was exact to all orders in the $1/N$ expansion, see also [11]. The result for the energy E_0 that we get from the ladder diagrams (3.1) does not agree precisely with the the gravity answer but it has the same dependence on the 't Hooft coupling [6].

In fact this model shows a transition at $\hat\lambda_c = 1/4$ between a single bound state and many bound states [9]. For very small $\hat\lambda$ we can approximate the potential by a delta function [6], which has only one bound state and reproduces (2.1). On the other hand, for very small energies $|E| \ll 1/L$ and for $L \ll |t| \ll 1/|E|$ we can solve the equation (3.1) by setting $\psi \sim |t|^\alpha$, with

$$\alpha(\alpha-1) + \hat\lambda = 0\,, \qquad \Rightarrow \qquad \alpha = \frac{1}{2} \pm i\sqrt{\hat\lambda - \frac{1}{4}}\,. \tag{3.2}$$

We see that for $\hat{\lambda} > \hat{\lambda}_c = 1/4$ the wave function oscillates as

$$\cos\left(i\left|\alpha - \frac{1}{2}\right|\log|t|\right) \ , \qquad \text{for} \quad L \ll |t| \ll 1/|E| \, . \tag{3.3}$$

We expect a turning point at around $|t_{turn}| \sim 1/|E|$. Since the equation is scale invariant in this region, the details of the turning point are E independent. By taking a small value of E we can get many zeroes of the cosine (3.3) before getting to the turning point where (3.3) ceases to be valid. This implies that we have an infinite number of bound states for $\lambda > \lambda_c$. Furthermore, the number of zeroes of (3.3) between the turning points at $t = \pm t_{turn}$ gives us the number of states below the energy we are considering. Thus the number of states goes as

$$n \sim 2\frac{\sqrt{\hat{\lambda} - \frac{1}{4}}}{\pi}\log|t_{turn}| \sim -2\frac{\sqrt{\hat{\lambda} - \frac{1}{4}}}{\pi}\log(-EL) \sim -\frac{\sqrt{g^2N}}{\pi^2}\log(-EL) \tag{3.4}$$

where in the last relation we assumed $g^2N \gg 1$. This precisely reproduced the string theory answer (2.21). It would be interesting to find if there is a rationale behind this agreement, since the ladder model includes only a small subset of all planar diagrams.

On the other hand, for $\hat{\lambda} < \hat{\lambda}_c$, there is only one bound state solution of (3.1). To show this, we set $E = 0$ in (3.1) and solve the equation in terms of hypergeometric functions. We then find that the solution with the right boundary conditions at $t = -\infty$ has a single zero for $\hat{\lambda} < 1/4$ (but it does not obey the right boundary conditions at $t = +\infty$). Therefore, there is a single bound state for (3.1) in this region.[4]

Is there any sign, on the string theory side, of a transition at a critical value of the radius? A precise answer to this question would require a solution of the string theory on $AdS_5 \times S^5$. We will content ourselves with the following qualitative argument. Let us consider again the folded string discussed above. Analyzing the string theory more precisely, we find that the tip of the string is better modelled by a massive particle. We can think of this massive particle as a massive string state or a Kaluza Klein state on $AdS_5 \times S^5$. So let us repeat the analysis of (2.19) for a massive particle at the tip of the string. The action is ($T = 1/(2\pi\alpha')$)

$$S = \int dt \left[-mR\frac{1}{z}\sqrt{1 - \dot{z}^2} + 2R^2T\frac{1}{z} \, .\right] \tag{3.5}$$

Now we write the energy conservation condition

$$\frac{mR}{z}\frac{1}{\sqrt{1 - \dot{z}^2}} - 2R^2T\frac{1}{z} = -2R^2T\frac{1}{z_M} + \frac{mR}{z_M} = E \, , \tag{3.6}$$

which allows us to solve for the motion of the tip position z. Let us imagine that the tip starts from rest at $z = z_M$. For $m < 2RT$, the tip falls towards the boundary (small z). For $z \ll z_M$, it reaches the asymptotic velocity given by

$$\sqrt{1 - \dot{z}^2} = \frac{m}{2TR} \, . \tag{3.7}$$

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[4]The two solutions to (3.1) with $E = 0$ are $F[\alpha_-, \alpha_+, \frac{1}{2}; -x^2]$, $xF[-\alpha_-, -\alpha_+, \frac{3}{2}; -x^2]$ where $\alpha_\pm = -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 - 4\hat{\lambda}}$.

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For $m \ll 2TR$, the asymptotic speed is very close to the speed of light. This happens in the case that $R \gg \sqrt{\alpha'}$, since m is at most $1/\sqrt{\alpha'}$ for a folded string. On the other hand, if the mass approaches $2TR$, then the final velocity becomes non-relativistic, and the approximations we have made around (2.19) are not valid. In fact, as we reduce the value of R and imagine that the tip of the string stays with a mass of order $1/\sqrt{\alpha'}$, we see that negative energy solutions of (3.6) cease to exist. In fact, for $m > 2TR$ the tip falls toward the horizon (large z), causing a disintegration of the string. Hence, for $R \ll \sqrt{\alpha'}$ the string cannot support any bound states. This provides a signal, from the dual AdS point of view, of a transition similar to the transition at $\hat{\lambda} = 1/4$ in the ladder diagrams. Of course, this argument is only qualitative. We will have to wait till the $AdS_5 \times S^5$ string theory is properly quantized in order to study the transition on the AdS side more precisely.

Another situation where this transition happens is the following. We now stay at strong coupling but consider different scalar charges for the quark and antiquark. In that case the string ends at two different points on S^5. In appendix A we display the solution describing the asymptotic motion of such a string. In order to get a qualitative idea we can model this situation by assuming that the mass at the tip is of the order of $m \sim TR$ up to a function of the angle on S^5. The case where (3.7) is zero corresponds to the case where the quark and anti-quark have opposite scalar charges. In this case the configuration is BPS and there is a family of static string solutions found in [12].

A third situation that can be qualitatively modelled by (3.5) is the one considered in [13]. They introduce a heavy quark anti-quark pair separated by some distance L and add angular momentum J along an $SO(2) \subset SO(6)$. This corresponds to binding J adjoint scalars to the heavy quark anti-quark pair. If J is very small, this corresponds to exciting some small transverse fluctuations of the string described around (2.11). When J is large we can model the situation by a heavy particle at the tip, of mass $m \sim J/R$, up to a numerical coefficient which depends on the details of the shape of the deformed string. The bound states disappear for $m \sim TR$, i.e., $J \sim TR^2 \sim \sqrt{g^2 N}$ (the precise numerical coefficient was computed in [13]). Note that for weak coupling there are no bound states.

Although the ladder approximation is obviously inadequate for large 't Hooft coupling, it is encouraging that its strong coupling limit is qualitatively similar to the exact AdS/CFT results. The lowest energy eigenvalue has the $-\sqrt{\lambda}/L$ coupling dependence in both cases although the dimensionless coefficients disagree for obvious reasons [7]. Furthermore, in both cases the gap between the lowest energy state and the continuum is populated with discrete levels, and the level spacing of the near threshold states matches exactly at strong coupling with strings containing one long fold. The big qualitative difference remaining is a discrepancy in the density of states far below the threshold, near the ground state. In the AdS/CFT case the levels are those of a string, namely those of an infinite number of oscillators with frequencies ω_n, $n = 1, 2 \cdots$. In contrast, the sum of ladder diagrams predicts discrete levels of a single harmonic oscillator. Presumably, including more complicated planar diagrams, which include, for example, long chains of gluons with near neighbor interactions [14], will remove this discrepancy.

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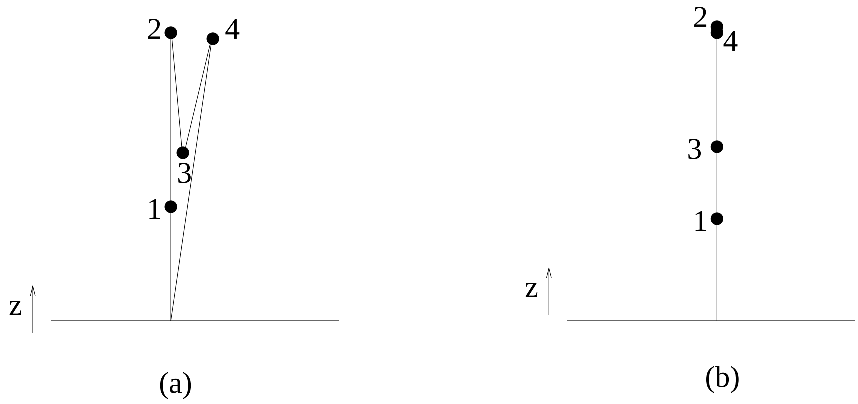

Figure 3: In (a) we show a configuration of a string stretched in the z direction with pieces along the string carrying momentum along the z direction. To facilitate visualization we have separated the strings in a transverse dimension, but we should think of all of the pieces as lying on top of each other as in (b).

Finally, we ask if the transition discussed above could also arise in QCD. In a naive ladder diagram approximation, we only have the gauge field exchange, but no scalar exchange. We then seem to find that the new bound states appear at

$$\frac{N\alpha_s}{\pi} > \frac{1}{2}, \quad \text{for QCD;} \qquad \frac{N\alpha_s}{\pi} > \frac{1}{4}, \quad \text{for } \mathcal{N}=4\,. \tag{3.8}$$

For $N = 3$ this gives $\alpha_s^c = \pi/6 \approx 0.5$. This is not an exact result in QCD, but rather an order of magnitude estimate. Note that a flux tube in a pure glue theory without dynamical quarks cannot exhibit a sudden appearance of states: because of confinement, the spectrum is completely discrete, without a threshold. Lattice studies indicate some crossover of gluonic excitation levels between short string and long string regimes, at string length around 1 fm [15]. However, there is no evidence for states with a $-1/L$ behavior of energy, other than the ground state (see of [15, figure 2]). Thus, it appears that in QCD the effects of asymptotic freedom do not allow us to see the particular transition we discussed in this article. Perhaps a simple qualitative way to explain this is to note that the coefficient of the $1/r$ potential remains small until the confining potential, linear in r, takes over.

4. Strings with many folds

In this section we consider the dynamics of strings with many folds. We consider the strict $L = 0$ limit. In order to do this we note that the whole motion of the string is taking place in an AdS_2 subspace of AdS_5 parametrized by the coordinates $t,\ z$. Actually, the problem has a natural $SO(2,1) \times SO(3)$ symmetry. As in [16] it is convenient to think of the gauge theory as defined on $AdS_2 \times S^2$. This is related by a simple Weyl transformation to $R^{1,3}$. In this case it is useful to parametrize AdS_5 as

$$ds^2 = d\rho^2 + \cosh^2\rho\, ds^2_{AdS_2} + \sinh^2\rho\, ds^2_{S^2}\,. \tag{4.1}$$

The simplest BPS Wilson loop associated to a single quark insertion corresponds to a string worldsheet located at $\rho = 0$. The folded string configuration we want to consider is also at $\rho = 0$. In a semiclassical approximation, we can consider n massless particles that are joined by stretched strings [10]. Of course, the "massless particles" are the parts of the string that are not stretched and are carrying momentum. The potential energy due to a stretched string segment going between two particles is

$$V = \alpha \left| \frac{1}{z_1} - \frac{1}{z_2} \right| , \qquad \alpha = R^2 T = \frac{\sqrt{g^2 N}}{2\pi} \tag{4.2}$$

We now consider a quark at the boundary at $z_0 = 0$ and a set of n massless particles joined by strings and ending at the boundary at $z_{n+1} = 0$.

$$H = \sum_{i=1}^{n} |p_i| + \sum_{i=0}^{n} V(z_i, z_{i+1}) \, . \tag{4.3}$$

It is interesting to compare this Hamiltonian with the naive Hamiltonian that we would get for n massless gluons that are in spherically symmetric wavefunctions and interacting via Coulomb exchange at weak coupling, with the identification $r_i \sim z_i$. This would give a Hamiltonian precisely of the same form, except that the coefficient α in (4.2) would be different, it would be $\alpha_{weak} = g^2 N/4\pi$. Note that for $\alpha \ll 1$ a Hamiltonian such as (4.3) would not be a good classical approximation to the quantum problem, while it is a good approximation if $\alpha \gg 1$. Note that we can think of the problem as follows. After a Kaluza Klein reduction on the S^2, the resulting theory is roughly like a two dimensional QCD theory with adjoint matter on AdS_2. It is amusing that the ordinary Coulomb potential looks confining in AdS_2. The Hamiltonian (4.3) is reminiscent of that found in the 2-d adjoint QCD [17]. To be more precise, we have to include processes where a pair of adjacent folds is created or destroyed. Such particle number changing processes are also present in the adjoint QCD.

5. Wilson lines in the adjoint representation

The $L \to 0$ limit of the quark anti-quark potential is also relevant for the analysis of the Wilson line in the adjoint representation. This relation to the quark and anti-quark Wilson lines shows that the adjoint Wilson line is not locally BPS saturated. What we find here is that this Wilson line operator has rather peculiar properties at strong coupling.

At weak coupling nothing dramatic happens. There are of course the usual UV divergences from self-energy corrections to the adjoint external source represented by the Wilson line, but these can always be renormalized. The divergence can be absorbed into a renormalization of the mass of the external adjoint source. After subtracting this divergence the spectrum of excitations, at weak coupling, is bounded from below. Since the external adjoint exerts an attractive force on the dynamical gluons, we need to use the uncertainty principle argument discussed in section 2.1 in order to show that gluons will not lead to very negative energy bound states at weak coupling.

On the other hand, according to the AdS/CFT correspondence, at large N and strong 't Hooft coupling the adjoint Wilson line introduces a folded string as shown in figure 2(c). Such a string has infinite energy since it stretches all the way to the boundary. This is not surprising; it also happens for the Wilson loop in the fundamental. This is a UV divergence which can be subtracted as in [3]. Once we subtract this infinite part, we can further decrease the energy by moving the tip of the fold towards the boundary. So, after this subtraction, the Hamiltonian in this sector is unbounded below, and any physical computation will involve a scattering amplitude where the fold comes in from the boundary and goes back to the boundary. Note that this feature of the Hamiltonian has real physical meaning and it is not due to a wrong renormalization procedure. This is the AdS dual of the "fall to the center" screening of the external adjoint source by a dynamical gluon when the coupling is sufficiently large. The situation is similar to that encountered in the two-dimensional string theory, dual to large N matrix quantum mechanics [18].[5] This transition is related to the fact that the linearized wave equation for a gluon in the field of an adjoint source displays a "fall to the center" instability [5].[6] In other words, we can interpret the tip of the folded string in figure 2(c) as a gluon or group of gluons in an S-wave and the straight string pieces as coming from the Coulomb interaction. The fact that the tip falls all the way to the boundary is then related to the gluon falling to the center. This peculiar behavior is also related to the fact that "quarkonium" is very small in theories with a gravity dual [19, 13].

As the coupling is reduced in the AdS description, the discussion below (3.6) shows that, below some critical coupling the tip of the fold falls towards the horizon rather than towards the boundary. This agrees with the absence of the "fall to the center" instability at weak coupling. In real world QCD, since the coupling decreases at short distances, there is no "fall to the center" instability; see the discussion after (3.8).

6. Correlation functions of Wilson loops and local operators

In this section[7] we will consider local vertex operators $\mathcal{O}$ and we will compute the expectation values of these vertex operators in the presence of the Wilson loop corresponding to a static quark and antiquark separated by distance L. Let us first consider an operator $\mathcal{O}_J = \mathrm{Tr}[\Phi^J] + \cdots$, where $\mathcal{O}_J$ is a BPS operator of dimension $\Delta = J$ in the $\mathcal{N} = 4$ theory. The first guess for the weak coupling scaling of such correlation functions is

$$\frac{\langle W\mathcal{O}(x)\rangle}{\langle W\rangle} \sim \frac{1}{x^{\Delta}} \tag{6.1}$$

at large x, where x is the distance of the operator insertion from the quark-antiquark pair, $x \gg L$.

[5] In this case we can do the computation on both sides of the duality and, after renormalization, we find a Hamiltonian that is unbounded below on both sides.

[6] We expect that, once we take into account the fields created by the gluon, sources in the fundamental representation will not cause a fall to the center instability for a gluon.

[7] This section was written in response to a question raised by E. Shuryak.

The strong coupling answer may be computed using the AdS/CFT duality as in [20]

$$\frac{\langle W\mathcal{O}(x)\rangle}{\langle W\rangle} \sim \frac{\sqrt{g^2N}}{N}\frac{L^{\Delta-1}}{x^{2\Delta-1}}\,. \tag{6.2}$$

The factor $\sqrt{g^2N}$ originates from R^2/α' in the string action. This result holds for any operator dual to a supergravity field, up to a numerical factor, which could be zero for some operators due to symmetry reasons.

Note that the fall-off as a function of x in (6.2) is faster for $\Delta > 1$. For specific operators, such as $\mathcal{O}'_4 = \mathrm{Tr}[F^2_{\mu\nu}]$, the angular dependence also differs from the naive weak coupling expectation that the expectation value of $\mathrm{Tr}\,E^2$ is given by the square of the dipole electric field $\sim L^2(1+3\cos^2\theta)/x^6$. The latter answer would be obtained in the $U(1)$ gauge theory, but the large N theory is very different from it due to the planarity restriction. In fact, we will show that the correct *planar* weak coupling answer has the same dependence on x as the strong coupling result (6.2).

The crucial difference between the $U(1)$ and the planar calculations is the following. For the $U(1)$ calculation the Coulomb exchanges between the charges play no role. We find field propagators connecting the operator insertion and the quark lines, see figure 4(a). The $U(1)$ answer, such as (6.1), arises from integrating the end-points of such propagators all the way along the time direction. If x is large, then the result (6.1) has contributions from the end points of the propagators on the quark lines separated by distances $\Delta t \sim x$ from each other.

At large N, the situation is different because the gluon exchanges between the quark and anti-quark keep changing their colors, but the pair of gluon fields emanating from O_4, for example, has correlated colors. For this reason, the positions of these lines cannot be integrated along the entire t-direction. The correct answer takes into account the fact that these gluon propagators, if placed at very different times along the quark and anti-quark lines, would remove the Coulomb attractive energy, much in the same way as what we discussed for gluon radiation in section 2. At weak coupling, the gluons that contribute to the Coulomb interaction energy are separated by a distance of the order of $\Delta t \sim 1/E_0 \sim L/(g^2N)$. Thus, the endpoints of the gluons connecting the operators to the quark lines should all end within a distance of the order of Δt of each other. If the gluon endpoint separations are bigger than Δt, we should get an exponential suppression due to the loss of the Coulomb binding energy. Thus, for distances $L/(g^2N) \gg x \gg L$ the large N weak 't Hooft coupling answer agrees qualitatively with the $U(1)$ intuition. But for $x \gg L/(g^2N)$ we find the same scaling as in (6.2).

Our conclusion is that, for distances $x \gg L/(g^2N)$, we can replace the endpoints of all propagators on the quark lines (suitably integrated) by a series of local color-singlet operators, see figure 4(c). Similarly, we can have graphs where the pair of gluons emanating from $\mathcal{O}$ is inserted on the same gluon line in the ladder graph. We can think of all the relevant graphs as propagation of a color-singlet state from the location of $\mathcal{O}$ to an insertion somewhere on the QCD string worldsheet. Such a description through color singlet diagrams is similar to what one does on the string side of the AdS/CFT correspondence.

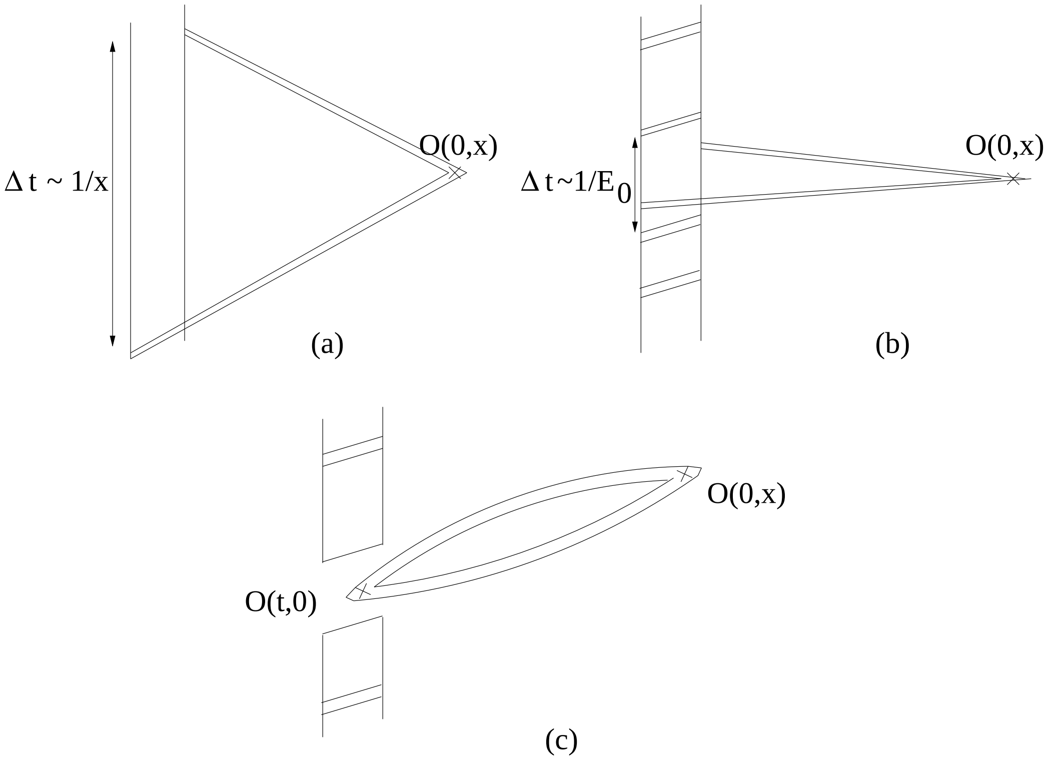

Figure 4: In (a) we see the diagrams contributing to the naive answer. It contains diagrams where the propagator endpoints are separated by a distance of the order of $\Delta t \sim 1/x$. In (b) we take into account the diagrams contributing to Coulomb energy between the quark and antiquark. Planarity plus energetic considerations effectively restrict the endpoints of the propagators to be within $\Delta t \sim 1/E_0$. In (c) we replace the square in the center of (b) by the insertion of a series of operators which is also integrated along t, the direction of the original contour.

We emphasize that for sufficiently large y this picture is valid at both weak and strong coupling of the large N gauge theory.

At this point we can replace a narrow Wilson loop by a series of color singlet operator insertions. The external operator $\mathcal{O}$ will give a non-trivial two point function only if $\mathcal{O}$ itself appears in this OPE. Let us consider first the operators $\mathcal{O}_J \sim \mathrm{Tr}[\Phi^J]$. Normalizing the two point functions of $\mathcal{O}_J$ to 1, the relevant term in the Wilson loop expansion will have the form

$$c_w \int dt \mathcal{O}_J \, , \qquad c_w \sim \frac{L^{J-1}}{(g^2N)^{J-1}} (g^2N)^{J/2} \frac{1}{N} \, , \tag{6.3}$$

where the integral is along the Euclidean time t and the operators are inserted at $x_1 = 0$ (recall that the quark and antiquark are located at $x_1 = \pm L/2$ respectively). The first factor in c_w (6.3) comes from integrating the endpoints of the propagators over a restricted range, see figure 4(b). The second comes from the normalization of the operator and powers of N that can be counted by looking at the difference between figures 4(b) 4(c).

Then the correct planar answer at weak ‘t Hooft coupling is

$$\frac{\langle W \mathcal{O}(x) \rangle}{\langle W \rangle} \sim c_w \int dt \langle \mathcal{O}_J(t,0) \mathcal{O}_J(0,x) \rangle = c_w \int dt \frac{1}{(t^2 + x^2)^{\Delta}} \sim c_w \, \frac{1}{x^{2\Delta - 1}} \tag{6.4}$$

where c_w is the constant appearing in (6.3). This result is correct for distances $x \gg L/(g^2N)$. Remarkably, this has the same structure as the strong coupling answer (6.2), except that the power of g^2N is different. We can reproduce the dependence of (6.2) on the 't Hooft coupling if we assume that the propagator endpoints are localized within a distance of the order of $\Delta t \sim L/\sqrt{g^2N}$, which is indeed the distance at which the strong coupling Coulomb potential becomes important.

In particular, this reproduces the result of [4] for the operator $\langle W \,\mathrm{Tr}[F_{\mu\nu}^2]\rangle$, $\sim L^3/x^7$. At weak coupling one might naively apply the $U(1)$ intuition suggesting the behavior as a square of a dipole field $L^2(1 + 3\cos^2\theta)/x^6$ where θ is the angle between $\vec{x}$ and the line between the quark-antiquark pair. Our arguments show that the correct weak coupling result at large N and large enough x has the same isotropic behavior as the strong coupling result of [4]. These arguments are also valid for the stress tensor. So the expectation value for the energy T_{00} also behaves as L^3/x^7 both at weak and strong coupling.

Finally, note that there may be $1/N$ suppressed contributions coming from non-planar diagrams, which decay more slowly than (6.4) at long distances. Thus for distances x which are parametrically large in N these contributions will dominate. Similar contributions could be present at strong coupling also. Presumably they come from loop diagrams in AdS, but we have not found their precise x dependence.

7. Discussion

One of the lessons from this article is that, already at weak coupling, large N effects produce some results that are in rough qualitative agreement with the strong coupling results provided by the AdS/CFT correspondence. One example is the existence of a gap for excitations of the flux tube connecting a heavy quark anti-quark pair. The second is the large x behavior of one point functions.

On the other hand, some features appear only at strong coupling; for example, the existence of an infinite number of bound state excitations of the flux tube. In this article we studied excited states of this flux tube near the "ionization" threshold. These states correspond to long folded strings in AdS. The density of states, computed with the ladder model, matched precisely the strong coupling answer based on a string with a single fold. Furthermore, the dynamics of a string with many folds resembles quite closely the expected dynamics of S-wave gluons in the gauge theory.

We might ask whether the transition we discussed here is evidence for a quantum phase transition in $\mathcal{N} = 4$ Yang Mills theory as a function of g^2N. We believe that the answer is no. In fact, perfectly analytic systems like ordinary quantum mechanical systems can display transitions in the structure of the spectrum without there being a phase transition. In fact, we also found that some of the strong coupling features, such as the long distance behavior of one point functions, are also present at weak coupling.

Finally, we mention that QCD flux tube excitations may play a role in sufficiently excited mesons (although, as mentioned above, these excitations occur in the linear confining regime of the flux tube). For example, some of the new mesons recently observed above

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the $D\bar{D}$ threshold in the $\mathbf{c\bar{c}}$ system (see [21] for a review) could be gluonic excitations of charmonium. In other words, perhaps some of these new mesons can be interpreted as hybrid states containing extra gluons.

Acknowledgments

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A. Solutions for strings ending at different points on S^5

We present a time dependent solution which describes strings sitting at a spatial point on the boundary but ending at two points on S^5 separated by some angle $\Delta\theta$. We have a string on the space

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} + d\theta^2 \tag{A.1}$$

We write the Nambu action in variables where $t = \tau$ and $\theta = \sigma$

$$\int dt d\theta \sqrt{\frac{1 - \dot{z}^2}{z^2} + \frac{z'^2}{z^4}} \tag{A.2}$$

where the prime denotes the derivative with respect to θ. We now guess a solution with a simple time dependence of the form $z = \tau f(\theta)$. The equation for f can be integrated once and we find

$$\frac{1 - f^2}{\sqrt{(1 - f^2) f^2 + f'^2}} = \frac{\sqrt{1 - f_0^2}}{f_0} , \quad \text{or} \quad \frac{df \sqrt{1 - f_0^2}}{\sqrt{(1 - f^2)(f_0^2 - f^2)}} = d\theta \tag{A.3}$$

where $0 < f_0 < 1$ is the point where $f' = 0$, which we take to be at $\theta = 0$. Notice that $f_0 = \partial_\tau z|_{\theta=0}$ is the speed of the motion of the tip. The equation (A.3) determines $f(\theta)$. We find

$$\frac{\Delta\theta}{2} = \sqrt{1 - f_0^2} \int_0^{f_0} \frac{df}{\sqrt{(1 - f^2)(f_0^2 - f^2)}} = \sqrt{1 - f_0^2} K[f_0] \tag{A.4}$$

where K is an elliptic function (see [22]). This equation gives us the velocity for the motion of the tip in terms of $\Delta\theta$. For $\Delta\theta = \pi$ the velocity goes to zero.

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